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On Fundamental Theorems of Approximation Theory and Their Dual Versions

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INTRODUCTION

This paper is concerned with theorems of D. Jackson and S. N. Bernstein, and with more recent ones due to M. Zamansky and S. B. Stečkin. In Butzer and Scherer ([5], [6]) it was shown that the assertions of these four theorems are equivalent to each other for polynomials of best approximation in $C_{2\pi}$-space, as well as in an arbitrary Banach space X in the setting of the theory of intermediate spaces. Our aim here is to establish these theorems in the dual setting, and to develop a theory of best approximation in the dual Banach space X', assuming such a theory as known in X.

For this purpose, the fundamental results are first stated in $C_{2\pi}$-space in Theorem I, and then extended to Banach spaces in Theorem II. These results can be found with proofs in [5], [6], but the emphasis here lies in working out the basic concepts and ideas used when passing over from the $C_{2\pi}$ to the abstract setting. This has been carried out in greater detail than in [5], [6], especially the topological questions, in order to obtain, in Section 2, an approach to the dual theory which is as intuitive as possible and yet as rigorous as necessary. The central role in the proofs is played by Jackson and Bernstein type inequalities for best approximation and approximants, respectively, as well as by their connection with the K interpolation method of Peetre ([10], [11]). Thus, Section 1 of the paper is partly of expository character.

In Section 2 concepts leading to a definition of dual best approximation are developed in all detail. In this connection, four basic duality relations [(8)–(11)] are presented. Two of these [namely, (9) and (10)] had previously been used by Singer and Buck in another context. Dual Jackson and Bernstein type inequalities are studied. Our main result is Theorem III, the dual version

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of Theorem II. An application to approximation theoretical characterizations of Banach spaces of periodic distributions is given by Theorem IV, the dual version of Theorem I.

1. Problem and Results in Banach Spaces

Let $t_n(x)$ be a trigonometric polynomial of degree $\leqslant(n-1)$, $n \in \mathbb{N}$, $\mathbb{N}$ being the set of all nonnegative integers, and let T_n be the corresponding linear space, the space T_0 consisting of the constant zero. If $C_{2\pi}$ denotes the space of all continuous 2π-periodic functions with norm $\|f\|_{C_{2\pi}} = \max_{x\in[-\pi,\pi]} |f(x)|$, we call

$$E_n(f; C_{2\pi}) = \min_{t_n \in T_n} \|f - t_n\|_{C_{2\pi}} \qquad (n \in \mathbb{N})$$

the best approximation of $f \in C_{2\pi}$ by elements of T_n, and denote by $t_n^*(f) = t_n^*(f; x) \in T_n$ the polynomial of best approximation to f. In this context, the well-known theorem of Weierstrass states that

$$\lim_{n\to\infty} E_n(f; C_{2\pi}) = 0 \qquad (f \in C_{2\pi}).$$

Another fundamental theorem of approximation theory is the following

THEOREM I1. *Let $f \in C_{2\pi}$. The following assertions are equivalent to each other for* $0 \leqslant k < \theta < l$ $(k, l \in \mathbb{N})$:

$$\sup_{0<n<\infty} n^{\theta} \|f - t_n^*(f)\|_{C_{2\pi}} < \infty, \qquad \text{(Ia)}$$

$$f^{(k)} \in C_{2\pi} \quad \text{and} \quad \sup_{0<n<\infty} n^{\theta-k} \|f^{(k)} - t_n^{*(k)}(f)\|_{C_{2\pi}} < \infty, \qquad \text{(Ib)}$$

$$\sup_{0<n<\infty} n^{\theta-l} \| t_n^{*(l)}(f)\|_{C_{2\pi}} < \infty. \qquad \text{(Ic)}$$

The implication (Ia) $\Rightarrow$ (Ic) is due to M. Zamansky [17], and (Ia) $\Rightarrow$ (Ib), is due to S. B. Stečkin [14]. The implication (Ic) $\Rightarrow$ (Ia) was first shown by Butzer–Pawelke [3] for $L_{2\pi}$-space and then for $C_{2\pi}$ (as well as for $L_{2\pi}$, $1 \leqslant p < \infty$) by G. Sunouchi ([15], [16]). The theorem as a whole is to be found in Butzer–Scherer ([5], [6]). For extensions to the n dimensional torus, as well as to a general class of manifolds including compact Lie groups, see H. Johnen ([7], [8a]).

Let $\text{Lip}^* \alpha = \{f \in C_{2\pi} : \sup_{0<t<\infty} t^{-\alpha}\omega_2(f; t; C_{2\pi}) < \infty\}$ and $\text{lip}^* \alpha = \{f \in C_{2\pi} : \lim_{t\to 0+} t^{-\alpha}\omega_2(f, t; C_{2\pi}) = 0\}$ for $\alpha > 0$, where

$$\omega_l(f, t; C_{2\pi}) = \sup_{|h|\leqslant t} \| \Delta_h^l f(x)\|_{C_{2\pi}}$$

is the modulus of smoothness of order $1 \leqslant l \in \mathbb{N}$, with

$$\Delta_h^l f(x) = \sum_{r=0}^{l} (-1)^{l-r} \binom{l}{r} f(x + rh).$$

Then, in addition to Theorem I1, the approximation theorems of D. Jackson and S. N. Bernstein as well as their extensions by A. Zygmund state

THEOREM I2. *If $f \in C_{2\pi}$ and $\theta = \alpha + r$, where $r \in \mathbb{N}$ and $0 < \alpha < 2$, then assertion* (Ia) *is equivalent to*

$$f^{(r)} \in \mathrm{Lip}^* \alpha. \tag{Id}$$

It is of interest to generalize Theorems I 1, I 2 to arbitrary Banach spaces. This was one of the tasks of Butzer–Scherer ([4], [5], [6]). In this connection we mention two important inequalities which play a fundamental role in the proofs of the equivalence of the assertions (Ia)–(Id). The Jackson inequality asserts that

$$E_n(f; C_{2\pi}) \leqslant C(n+1)^{-r} E_n(f; C_{2\pi}^{(r)}) \leqslant C(n+1)^{-r} \|f\|_{C_{2\pi}^{(r)}} \qquad (f \in C_{2\pi}^{(r)}), \tag{$\mathrm{J}_{C_{2\pi}}$}$$

while the Bernstein inequality reads

$$\| t_n \|_{C_{2\pi}^{(r)}} \leqslant n^r \| t_n \|_{C_{2\pi}} \qquad (t_n \in T_n,\ n \in \mathbb{N}). \tag{$\mathrm{B}_{C_{2\pi}}$}$$

Here, $C_{2\pi}^{(r)}$ is the Banach subspace of $C_{2\pi}$ of all functions f for which $f^{(r)} \in C_{2\pi}$, with norm $\|f\|_{C_{2\pi}^{(r)}} = \|f\|_{C_{2\pi}} + \|f^{(r)}\|_{C_{2\pi}}$.

In order to formulate the desired results in the abstract setting, the preceding concepts must first be transcribed to Banach spaces. Thus, we replace the space $C_{2\pi}$ by an arbitrary Banach space X, and the sequence $\{T_n\}_0^\infty$ by a sequence of closed subspaces $\{P_n\}_0^\infty$:

$$P_0 = \{0\} \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots \subset X^1. \tag{1}$$

We can then define for every $f \in X$

$$E_n(f; X) = \inf_{p_n \in P_n} \|f - p_n\|_X \qquad (n \in \mathbb{N}),$$

and to obtain a closer connection between $\{P_n\}^\infty$ and X we demand, as a generalization of the theorem of Weierstrass,

$$\lim_{n\to\infty} E_n(f; X) = 0 \qquad (f \in X). \tag{W_X}$$

[1] The symbol "$\subset$", here and in the following, means that, algebraically, one space is contained in the other, and that, topologically, the identity map is continuous, i.e., in case of Banach spaces Y contained in X, $\|f\|_X \leqslant \|f\|_Y$ holds for all $f \in Y$. If $Y \subset X$ and $X \subset Y$ for any two Banach spaces, we shall write $X \cong Y$. If we want to emphasize a merely algebraic equality, we simply write $X = Y$.

Thus, $\bigcup_0^\infty P_n \equiv P$ is dense in X, and this is no essential restriction on X; for, otherwise, we could work with the closure of P in X. A subspace of smooth elements in X, such as $C_{2\pi}^{(r)}$ in $C_{2\pi}$, is given by an approximation space Y of X which satisfies by definition

$$P \subset Y \subset X. \tag{2}$$

In the following, we shall restrict ourselves to approximation spaces Y with

$$\lim_{n\to\infty} E_n(f; Y) = 0 \qquad (f \in Y). \tag{W_Y}$$

The space $C_{2\pi}^{(r)}$, for instance, satisfies (W_y), since it is topologically isomorphic to $C_{2\pi}$.

In order to connect the approximation concepts for X and for Y, we demand as a generalization of $(J_{C_{2\pi}})$ the Jackson type inequality of order $\sigma \geqslant 0$:

$$E_n(f; X) \leqslant C(n+1)^{-\sigma} E_n(f; Y) \qquad (f \in Y, n \in \mathbb{N}), \tag{J}$$

and, as a generalization of $(B_{C_{2\pi}})$, the Bernstein type inequality of order $\sigma \geqslant 0$:

$$\| p_n \|_X \geqslant D(n+1)^{-\sigma} \| p_n \|_Y \qquad (p_n \in P_n, n \in \mathbb{N}), \tag{B}$$

C, D being positive constants which depend only upon Y and σ. In these two relations, the measure of best approximation of $f \in Y$ by elements of P_n and the approximants p_n of P_n are compared in the different norms of the spaces X and Y; in the case of (J), this means a certain rate of decrease of $E_n(f; X)$ and in (B), a certain rate of increase of $\| p_n \|_X$.

A characterization of approximation spaces Y of order $\sigma \geqslant 0$, i.e., spaces Y satisfying (J) and (B), is given by (see [4], p. 61)

LEMMA 1. *Y is an approximation space of X of order $\sigma \geqslant 0$ if, and only if, for every $f \in Y$,*

$$C_\sigma \sup_{0<n<\infty} n^\sigma E_{n-1}(f; X) \leqslant \| f \|_Y \leqslant D_\sigma \sum_{n=1}^{\infty} [n^\sigma E_{n-1}(f; X)] \frac{1}{n},$$

where C_σ, D_σ are positive constants depending only upon C, D and σ.

Finally, we assume the existence[2] of an element $p_n^*(f; X) \in P_n$ of best approximation to f in X, i.e.,

$$\| p_n^*(f; X) - f \|_X = E_n(f X). \qquad (n \in \mathbb{N}). \tag{3}$$

[2] This is no essential restriction; otherwise, instead of $p_n^*(f; X)$, we could consider elements of the sets $P_{n,\sigma,M}(f; X) = \{p_n \in P_n : \| f - p_n \|_X \leqslant E_n(f; X) + Mn^{-\sigma} \| f \|_X\}$, which are nonempty. This is closely related to the notion of "good approximation" (compare Buck [1]).

Denoting by l_*^q the space of all sequences $\{a_n\}_1^\infty$ such that $\sum_1^\infty |a_n|^q(1/n) < \infty$ ($\sup_{1<n<\infty} |a_n| < \infty$, for $q = \infty$), we may state (see [6])

THEOREM II1. *If Y is an approximation space of X of order $\sigma \geqslant 0$, the following assertions are equivalent for $f \in X$ and $\theta > 0$, $1 \leqslant q \leqslant \infty$:*

$$\{n^\theta E_{n-1}(f; X)\} \in l_*^q, \tag{IIa}$$

$$f \in Y \qquad \{n^{\theta-\sigma} \| f - p_n^*(f; X)\|_Y\} \in l_*^q \qquad (\sigma < \theta) \tag{IIb}$$

$$\{n^{\theta-\sigma} \| p_n^*(f; X)\|_Y \in l_*^q \qquad (\theta < \sigma). \tag{IIc}$$

Since $E_n(f; X) \leqslant C(n+1)^{-\sigma} E_n(f; Y) \leqslant C(n+1)^{-\sigma} \| f - p_n^*(f; X)\|_Y$ for every $f \in Y$, it is an immediate consequence of Theorem II1 that the assertion

$$f \in Y, \qquad \{n^{\theta-\sigma} E_{n-1}(f; Y)\} \in l_*^q \qquad (\sigma < \theta) \tag{IIb*}$$

is equivalent to the assertions (IIa)–(IIc).
Theorem II1 is evidently a generalization of the theorems of S. B. Stečkin and M. Zamansky and their converses to Banach spaces. Indeed, in case $X = C_{2\pi}$, $Y = C_{2\pi}^{(k)}$ or $Y = C_{2\pi}^{(l)}$ and $P_n = T_n$, it specializes, for $q = \infty$, to Theorem I1. To obtain a generalization of assertion (Id), we replace the the modulus $\omega_l(f, t; C_{2\pi})$ by the functional (see Peetre [10])

$$K(f, t) \equiv K(f, t; X, Y) = \inf_{g \in Y} (\| f - g \|_X + t \| g \|_Y) \qquad (f \in X; 0 < t < \infty). \tag{4}$$

The K-functional $K(f, t)$ has properties similar to those of $\omega_l(f, t; C_{2\pi})$. Indeed, it is continuous and monotone decreasing in t, with $\lim_{t \to 0+} K(f, t) = 0$, for every $f \in X$, and it is a norm on X for each fixed $0 < t < \infty$. This is even clearer in view of the fact (see Butzer–Berens [2], pp. 192) that for $f \in C_{2\pi}$, $0 < \theta < l \in \mathbb{N}$,

$$\sup_{0<t<\infty} t^{-\theta}\omega_l(f, t; C_{2\pi}) < \infty \Leftrightarrow \sup_{0<t<\infty} t^{-\theta} K(f, t^l; C_{2\pi}, C_{2\pi}^{(l)}) < \infty, \tag{5a}$$

$$\lim_{t \to 0+} t^{-\theta}\omega_l(f, t; C_{2\pi}) = 0 \Leftrightarrow \lim_{t \to 0+} t^{-\theta} K(f, t^l; C_{2\pi}, C_{2\pi}^{(l)} = 0. \tag{5b}$$

$K(f, t)$ has the advantage that it is defined for each pair X, Y of Banach spaces with $Y \subset X$, while $\omega_l(f, t, C_{2\pi})$ is only defined for those spaces for which the concept of translation exists.

In view of (5a), the following is a generalization of the theorems of D. Jackson and S. N. Bernstein to Banach spaces.

THEOREM II2. *Let $f \in X$ and let $0 < \theta < \sigma$, $1 \leqslant q \leqslant \infty$. The assertion* (IIa) *is equivalent to*

$$\{n^\theta K(f, n^{-\sigma}; X, Y)\} \in l_*^q. \tag{IIa}$$

For a proof, see Peetre [10].

Finally, we formulate the above results in the terminology of the theory of intermediate spaces. To this end, we do not consider an individual element as in Theorem I, but gather all elements satisfying certain properties in a class. Thus, we define for $\theta > 0$, $1 \leqslant q \leqslant \infty$ (with obvious modifications for $q = \infty$):

$$X_{\theta,q} \equiv \Big\{ f \in X\colon \|f\|_{\theta,q;X} \equiv \Big(\sum_{n=1}^{\infty} [n^{\theta} E_{n-1}(f; X)]^q 1/n \Big)^{1/q} < \infty \Big\};$$

$$(X, Y)_{\theta/\sigma,q;K} \equiv \Big\{ f \in X\colon \|f\|_{\theta/\sigma,1;K} \equiv \Big(\sum_{n=1} [n^{\theta} K(f, n^{-\sigma}; X, Y)]^q 1/n \Big)^{1/q} < \infty \Big\};$$

$$(X, Y)^0_{\theta/\sigma,\infty;K} \equiv \{ f \in X\colon \|f\|_{\theta/\sigma,\infty;K} < \infty, \lim_{t\to 0+} t^{-\theta} K(f, t^{\sigma}; X, Y) = 0 \}.$$

These spaces are all Banach spaces (see, e.g., [4], [11]). From the equivalence of the assertions (IIa), (IIb*), and (IId) and from the relations (5a), (5b), it then follows that

$$X_{\theta,q} \cong Y_{\theta-\sigma,q} \cong (X, Y)_{\theta/\sigma/q;K}\,; \tag{6}$$

$$\mathrm{Lip}^* \,\alpha = (C_{2\pi}, C_{2\pi}^{(2)})_{\alpha/2,\infty;K}\,; \qquad \mathrm{lip}^* \,\alpha = (C_{2\pi}, C_{2\pi}^{(2)})^0_{\alpha/2,\infty;K}\,. \tag{7}$$

We remark that by the same methods not only does the algebraic equality of these spaces follow but also the topological equality. We conclude this section with a lemma about inclusion relations among these spaces; they follows almost by definition or by Lemma 1.

LEMMA 2. *The following inclusions hold*:

$$Y \subset (X, Y)_{\theta/\sigma,q;K} \subset X \qquad (0 < \theta < \sigma,\ 1 \leqslant q \leqslant \infty),$$

$$X_{\sigma,1} \subset Y \subset X_{\sigma,\infty}\,,$$

$$X_{\theta',q} \subset X_{\theta,q} \subset X_{\theta,p} \qquad (0 \leqslant \theta \leqslant \theta',\ 1 \leqslant q \leqslant p \leqslant \infty).$$

It is the first of these relations which leads one to speak of $(X, Y)_{\theta/\sigma,q:K}$ as intermediate spaces of X and Y.

2. THE DUAL PROBLEM

In the preceding section we extended Theorem I to Banach spaces X and Y, related by the properties (1), (2), (W_X), (W_Y), (J), and (B). This gave Theorem II. The aim of this section is a dual version of Theorem II, thus an analogous theorem for the spaces X', Y' of all bounded linear functionals defined on X, Y as described above, with

$$\|f'\|_{X'} = \sup_{0 \neq f \in X} |\langle f', f\rangle| / \|f\|_X .$$

In other words, we shall transfer the preceding theory in Banach spaces into its dual range.

To this end we develop further a concept of duality in approximation theory used by Singer [13], Buck [1], etc., in another context. We note that the two basic approximation theoretic notions of approximant and best approximation originate in the sequences $\{(P_n , X)\}_0^\infty$, $\{(P_n , Y)\}_0^\infty$ [3] and the associated sequences $\{X/(P_n , X)\}_0^\infty$, $\{Y/(P_n , Y)\}_0^\infty$ of quotient spaces because, e.g., $E_n(f; X)$ can be interpreted as the norm of the equivalence class $f + (P_n , X)$ in the quotient space $X/(P_n , X)$.

Now, it is well-known that the duals of (P_n , X), (P_n , Y) and $X/(P_n , X)$, $Y/(P_n , Y)$ can be identified by a natural isometric isomorphism with $X'/(P_n , X)^\perp$, $Y'/(P_n , Y)^\perp$ and $(P_n , X)^\perp$, $(P_n , Y)^\perp$, respectively. Here, $(P_n , X)^\perp$ denotes the annihilator of (P_n , X) in X', i.e., the space of all functionals of X' which vanish on (P_n , X). The norm of $X'/(P_n , X)^\perp$ is given by

$$E_n'(f'; X') \equiv \inf_{p_n' \in (P_n, X)^\perp} \|f' - p_n'\|_{X'} \qquad \text{for } f' \in X'.$$

These remarks lead to the following simple but basic relations:

$$\|p_n'\|_{X'} = \sup_{f \in X} \frac{|\langle p_n', f\rangle|}{E_n(f; X)} \qquad (p_n' \in (P_n , X)^\perp), \tag{8}$$

$$E_n'(f'; X') = \sup_{p_n \in P_n} \frac{|\langle f', p_n\rangle|}{\|p_n\|_X} \qquad (f' \in X'), \tag{9}$$

$$E_n(f; X) = \sup_{p_n' \in (P_n, X)^\perp} \frac{|\langle p_n', f\rangle|}{\|p_n'\|_{X'}} \qquad (f \in X), \tag{10}$$

$$\|p_n\|_X = \sup_{f' \in X'} \frac{|\langle f', p_n\rangle|}{E_n'(f'; X')} \qquad (p_n \in (P_n , X)), \tag{11}$$

the same relations also holding for Y', $(P_n , Y)^\perp$, and $E_n'(f'; Y')$. Relations (10), (11) are obtained by the canonical embedding of $X/(P_n , X)$ in $(X/(P_n , X))''$ and (P_n , X) in $(P_n, X)''$

[3] Here and in the following, it is necessary to distinguish between P_n as a normed subspace of X and P_n as a normed subspace of Y. We, therefore, write (P_n , X) or (P_n , Y), and remark that $(P_n , X) \cong (P_n , Y)$, in view of (2) and (B).

On the one hand, in view of the relations (8) and (10), the best approximation $E_n(f; X)$ is interchanged with $\| p_n' \|_X$; on the other hand, in view of (9) and (11), the norm of the approximant $\| p_n \|_X$ is interchanged with $E_n'(f'; X')$. Thus, we are led to a "dual" concept of approximation, with $E_n'(f'; X')$, $E_n'(f'; Y')$ and $(P_n, X)^\perp$, $(P_n, Y)^\perp$ replacing $E_n(f; X)$, $E_n(f; Y)$ and (P_n, X), (P_n, Y), respectively. However, the roles of best approximation and approximant are to be interchanged (in the sense just mentioned). The phenomenon in which concepts are interchanged and properties are reversed when passing over to the dual version (i.e., dual spaces or norms) is typical, and can, indeed, serve as a guide throughout this section.

Before applying the above notions in a dualized theory, some modifications and additions are needed. In view of Lemma 2, the Stečkin type assertion of Theorem II1 gave a characterization of the space $X_{\theta,q}$ in terms of a space Y (satisfying (J) and (B), with $\sigma > \theta$) with $X_{\theta,q} \subset Y$, while the Zamansky type assertion gave one in terms of (another, $\sigma > \theta$) space Y with $X_{\theta,q} \supset Y$. Consequently, a dual approximation concept must take into account the duals Y' of approximation spaces Y of various orders σ. Since the inclusion $Y \subset X$ is reversed to $X' \subset Y'$,[4] we shall construct a suitable space of functionals (to be denoted by D_X') which will contain these duals Y' (for all orders σ).

To this end, we consider the intersections D_X, D_Y of all approximation spaces $X_{\theta,q}$ and $Y_{\theta,q}$, respectively. In view of (6) and Lemma 2, we have $D_X = D_Y = \bigcap_{r=1}^{\infty} X_{r,\infty}$. If we endow D_X with the local convex topology generated by the family $\{\| f \|_{r,\infty;X}\}_0^\infty$ of seminorms of $f \in D_X$, it is easily shown that, under this topology, D_X is a Fréchet space which contains P and is dense in Y, by property (W_Y). If we endow the dual D_X' with the $\sigma(D_X', D_X)$-topology, we have for approximation spaces Y of all orders $\sigma \geqslant 0$,

$$X' \subset Y' \subset D_X'. \tag{12}$$

In view of $(P_n, X)^\perp \subset (P_n, Y)^\perp$ and the dual counterpart of (1), namely,

$$\cdots \subset (P_n, X)^\perp \subset \cdots \subset (P_0, X)^\perp \equiv X' \subset Y' \subset D_X',$$

we describe now approximation in the space D_X' in terms of a "dual best approximation." In the above discussion, we were led to the quantities $E_n'(f'; X')$, $E_n'(f'; Y')$. But, a statement like $\{n^\theta E_n'(f'; X')\} \in l_*^q$ [cf. IIb] yields trivialities since it either holds for all $f' \in X'(\theta < 0, 1 \leqslant q \leqslant \infty$ and $\theta = 0$, $q = \infty)$ or only for the zero element. To overcome this difficulty, we define the right side of (9) as the "dual best approximation" of f' (by elements of $(P_n, X)^\perp$), denoting it by $\| f' \|_{(P_n, X)}$, the norm of the restriction of $f' \in X'$ to (P_n, X). This quantity extends $E_n'(f'; X')$ from X' to D_X' since

[4] This inclusion is meaningful only if Y is dense in X; this, however, follows by (2) and (W_X).

it is also meaningful, i.e., finite, for f' in D_X'. This is a consequence of the fact that the topologies of D_X and X are equivalent on (P_n, X) for each fixed n, since $\| p_n \|_{r,\infty} = \sup_{0 \leqslant k \leqslant n} k^r E_k(p_n ; X) \leqslant n^r \| p_n \|_X$ for every $p_n \in (P_n, X)$ and $r \in \mathbb{N}$. Defining $\| f' \|_{(P_n, Y)}$ similarly, we can now deduce dual Jackson- and Bernstein type inequalities.

LEMMA 3. (a) *The Jackson inequality* (J) *holds if, and only if,* $(C' = C^{-1})$

$$\| p_n' \|_{X'} \geqslant C'(n + 1)^{\sigma} \| p_n' \|_{Y'} \qquad (p_n' \in (P_n, X)^{\perp}). \tag{B'}$$

(b) *The Bernstein inequality* (B) *holds if, and only if,* $(D' = D^{-1})$

$$\| f' \|_{(P_n, X)} \leqslant D'(n + 1)^{\sigma} \| f' \|_{(P_n, X)} \qquad (f' \in D_X'). \tag{J'}$$

Proof. The implication (J) $\Rightarrow$ (B') follows by applying (8) for $\| p_n' \|_{X'}$ and the corresponding relation for $\| p_n' \|_{Y'}$:

$$\| p_n' \|_{X'} = \sup_{f \in X} \frac{|\langle p_n', f \rangle|}{E_n(f; X)} \geqslant C'(n + 1)^{\sigma} \sup_{f \in Y} \frac{|\langle p_n', f \rangle|}{E_n(f; Y)} = C'(n + 1)^{\sigma} \| p_n' \|_{Y'} .$$

By an analogous conclusion, the implications (B′) $\Rightarrow$ (J), (B) $\Rightarrow$ (J′), and (J′) $\Rightarrow$ (B) follow, using the relations (10), (9), and (11), respectively.

Thus, a Jackson inequality for best approximation converts into a dual Bernstein inequality for the approximants and, conversely, a Bernstein inequality for approximants converts into a Jackson inequality for dual best approximation. The dual inequalities (B′), (J′) hold for the "small" space $(P_n, X)^{\perp}$ of approximating functionals and the "large" space D_X' of functionals to be approximated, while (B), (J) hold for the "large" (in the topological sense) space (P_n, X) of approximants and the "small" space Y of elements to be approximated. We further remark that the exponent in (B′), (J′) has a sign that is opposite of that in (B) and (J). In view of these "reversal" phenomena, in the following we shall consider dual approximation assertions such as $\{n^{-\theta} \| f' \|_{(P_n, X)}\} \in l_*^q$ with $\theta > 0$ (instead of $-\theta > 0$) for elements $f' \in D_X'$. This means that we consider functionals f' with a certain rate of increase of their dual best approximation, in contradistinction to (IIa), showing a certain rate of decrease of best approximation.

THEOREM III1. *Let* X, Y *be given as in Theorem* II *and let* $\theta > 0$, $1 \leqslant q \leqslant \infty$. *Then the following assertions are equivalent for* $f' \in D_X'$:

$$\{n^{-\theta} \| f' \|_{(P_n, X)}\} \in l_*^q, \tag{IIIa}$$

$$\{n^{-\theta} \| f'_{X,n} \|_{Y'}\} \in l_*^q \qquad (\sigma < \theta), \tag{IIIb}$$

$$f' \in Y', \quad \{n^{\sigma-\theta} \| f' - f'_{X,n} \|_{Y'}\} \in l_*^q \qquad (\sigma > \theta). \tag{IIIc}$$

Here, $f'_{X,n}$ *is a functional of* X' *satisfying, for each* $p_n \in P_n$,

$$\langle f'_{X,n}, p_n\rangle = \langle f', p_n\rangle; \qquad \| f'_{X,n} \|_{X'} = \| f' \|_{(P_n, X)}. \tag{13}$$

Such a functional $f'_{X,n}$ exists by the extension theorem of Hahn–Banach since the restriction of $f' \in D_X'$ is a continuous linear functional on (P_n, X) with finite norm $\|f'\|_{(P_n,X)}$. If $f' \in X'$, then the functional $p_n^{*\prime}(f'; X') = f' - f'_{X,n}$ belongs to $(P_n, X)^\perp$ and satisfies

$$\| f' - p_n^{*\prime}(f'; X')\|_{X'} = \| f' \|_{(P_n,X)} \equiv E_n'(f; X').$$

Thus, $f'_{X,n}$ plays the role of $f - p_n^*(f; X)$, and (IIIb), (IIIc) are assertions of Stečkin and Zamansky type, respectively, for $p_n^{*\prime}(f'; X')$. But according to the "reversal" phenomenon, here $f'_{X,n}$ is "smooth", while, originally, the element $p_n^*(f; X)$ of best approximation was "smooth". Thus, we could also call (IIIb), (IIIc) assertions of Zamansky and Stečkin type, respectively, for $f'_{X,n}$. In this sense, Zamansky and Stečkin type assertions are dual to each other just as are (shown above) the Jackson and Bernstein type inequalities. We remark further that, as in Theorem II1, we can prove a reduction type assertion; namely,

$$\{n^{\sigma-\theta} \| f' \|_{(P_n, Y)}\} \in l_*^q, \tag{IIIb*}$$

which follows immediately from the above theorem, since $\|f'_{X,n}\|_{Y'} \geqslant \|f'\|_{(P_n,Y)} \geqslant D(n+1)^{-\sigma} \|f'\|_{(P_n,X)}$.

Proof of Theorem III1. The proof is quite analogous to that of Theorem II in view of our approach. For technical reasons, we set $P_t = P_{[t]}$, $f'_{X,t} = f'_{X,[t]}$, for $0 < t < \infty$, and $\epsilon_k = 2^{\pm k}$, $\eta_k = 2^k$, $\delta_k = 2^{-k}$.

Then by (13), the Bernstein inequality (B′) yields

$$\begin{aligned} \| f'_{X,t\epsilon_{k+1}} - f'_{X,t\epsilon_k} \|_{Y'} &\leqslant C'(t\epsilon_k)^{-\sigma} \| f'_{X,t\epsilon_{k+1}} - f'_{X,t\epsilon_k} \|_{X'} \\ &\leqslant 2 \cdot 2^\sigma C'(t\epsilon_k)^{-\sigma} \| f' \|_{(P_{2t\epsilon_k}, X)}. \end{aligned} \tag{14}$$

A converse is given by the Jackson inequality (J′):

$$\begin{aligned} \| f' \|_{(P_t,X)} &\leqslant \sum_{k=0}^\infty [\| f' \|_{(P_{t\delta_k}, X)} - \| f' \|_{(P_{t\delta_{k+1}}, X)}] \\ &\leqslant \sum_{k=0}^\infty \| f'_{X,t\delta_k} - f'_{X,t\delta_{k+1}} \|_{(P_{t\delta_k}, X)} \\ &\leqslant 2^\sigma \sum_{k=0}^\infty \max(1, (t\delta_k)^\sigma) \| f'_{X,t\delta_k} - f'_{X,t\delta_{k+1}} \|_{Y'}. \end{aligned} \tag{15}$$

Now the implications (a) $\Rightarrow$ (b), (a) $\Rightarrow$ (c) follow from (14) (case + and − of $\epsilon_k = 2^{\pm k}$, respectively) and (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a), from (15), by a purely technical device (see [6]). As an illustration, we choose the most complicated implication, namely (a) $\Rightarrow$ (c).

We have first to show that $f' \in Y'$. Taking $t = 1$ in (14), (a) yields, in case $\sigma > \theta$,

$$\sum_{k=-1}^{\infty} \| f'_{X,\eta_{k+1}} - f'_{X,\eta_k} \|_{Y'} \leqslant 2 \cdot 2^{\sigma} C' \sum_{k=-1}^{\infty} 2^{-k\sigma} \| f' \|_{(P\eta_{k+1},X)}$$

$$\leqslant 2 \cdot 4^{\sigma} [\sup_{0<k<\infty} k^{-\theta} \| f' \|_{(P_k,X)}] \sum_{k=0}^{\infty} 2^{k(\theta-\sigma)} < \infty.$$

Thus, $\sum_{k=-1}^{\infty} [f'_{X,\eta_{k+1}} - f'_{X,\eta_k}]$ converges to an element of Y'. Since $\langle \sum_{k=-1}^{\infty} [f'_{X,\eta_{k+1}} - f'_{X,\eta_k}], p_{\eta_n} \rangle = \langle f'_{X,\eta_n}, p_{\eta_n} \rangle = \langle f', p_{\eta_n} \rangle$ for every $p_{\eta_n} \in P_{\eta_n}$ and $n \in \mathbb{N}$, and since P is dense in Y, it follows that $f' \in Y'$.

Then (14) and the generalized Minkowski inequality give

$$\left\{ \int_0^{\infty} [t^{\sigma-\theta} \| f' - f'_{X,t} \|_{Y'}]^q \, dt/t \right\}^{1/q}$$

$$\leqslant \left\{ \int_0^{\infty} \left[t^{\sigma-\theta} \sum_{k=0}^{\infty} \| f'_{X,t\eta_{k+1}} - f'_{X,t\eta_k} \|_{Y'} \right]^q dt/t \right\}^{1/q}$$

$$\leqslant 2 \cdot 2^{\sigma} C' \sum_{k=0}^{\infty} 2^{-k\sigma} \left\{ \int_0^{\infty} [t^{-\theta} \| f' \|_{(P_{t\eta_{k+1}},X)}]^q \, dt/t \right\}^{1/q}$$

$$\leqslant 2 \cdot 4^{\sigma} C' \left\{ \int_0^{\infty} [t^{-\theta} \| f' \|_{(P_t,X)}]^q \, dt/t \right\}^{1/q} \sum_{k=0}^{\infty} 2^{-k(\theta-\sigma)}.$$

Observing that $P_t = P_{[t]}$ and $f'_{X,t} = f'_{X,[t]}$, the integrals on both sides can be replaced by corresponding sums; thus, (c) follows from (a).

We now prove Jackson's and Bernstein's theorem for bounded linear functionals, which is the counterpart of Theorem II2.

THEOREM III2. *Let* X, Y *be given as in Theorem* II, *and let* $\theta > 0$, $1 \leqslant q \leqslant \infty$. *Then the assertions of Theorem* III1 *are equivalent to*

$$f' \in Y', \qquad \{n^{\sigma-\theta} K(n^{-\sigma}, f'; Y', X')\} \in l_*^{\,q} \qquad (\sigma > \theta). \tag{IIId}$$

Proof. Suppose (d) holds. Then for every representation $f' = f_1' + f_2'$, with $f_2' \in X'$, the inequality (J′) gives

$$\| f' \|_{(P_n,X)} \leqslant \| f_1' \|_{(P_n,X)} + \| f_2' \|_{(P_n,X)} \leqslant D'(n+1)^{\sigma} \| f_1' \|_{Y'} + \| f_2' \|_{X'} .$$

Taking the infimum over all possible representations, we have

$$n^{-\theta} \| f' \|_{(P_n,X)} \leqslant \max(D'2^\sigma, 1)\, n^{\sigma-\theta} K(n^{-\sigma}, f'; Y', X'),$$

for every $f' \in Y'$, and (a) follows by (12).

Conversely, by definition of the K functional and (13), we have, for each $f' \in Y'$,

$$n^{\sigma-\theta} K(n^{-\sigma}, f'; Y', X') \leqslant n^{\sigma-\theta} \| f' - f'_{X,n} \|_{Y'} + n^{-\theta} \| f' \|_{(P_n,X)},$$

whence (IIId) follows from (IIIa) plus (IIIc).

Finally, let us consider the duals in relation (6), namely,

$$(X_{\theta,q})' \cong (Y_{\theta-\sigma,q})' \cong ((X, Y)_{\theta/\sigma,q;K})'. \tag{6$'$}$$

By the duality theorem for K-intermediate spaces (see [11], [12]),

$$((X, Y)_{\theta/\sigma,q;K})' \cong (Y', X')_{1-\theta/\sigma,q';K} \qquad (1/q' + 1/q = 1,\ 1 \leqslant q < \infty),$$

$$((X, Y)^0_{\theta/\sigma,\infty;K})' \cong (Y', X')_{1-\theta/\sigma,1,K} \qquad (q = \infty). \tag{16}$$

Since the assertion (IIId) reads $f' \in (Y', X')_{1-\theta/\sigma,q';K}$, with q replaced by q', Theorem III2 and (6)$'$ show that the assertions (IIIa)–(IIIc) characterize the duals of the spaces which occur in Theorem II1. This is a final justification for speaking of Theorem III as the dual of Theorem II.

With the notations of this theorem, we take $X = C_{2\pi}$, $Y = C_{2\pi}^{(r)}$, and $P_n = T_n$. Then X' is the space $BV_{2\pi}$ of all 2π-periodic functions of bounded variation, $D_X{}'$ becomes the space D of all 2π-periodic distributions, and $(C_{2\pi}^{(r)})' = \{f' \in D : f = g^{(r)}, g \in BV_{2\pi}\} = BV_{2\pi}^{(-r)}$, where the derivatives are to be understood in the distributional sense. To explain the last notation, we define the r-th integral of a distribution $f' \in D$ by its Fourier series:

$$f' = \sum_{k=-\infty}^{\infty} f'^{\wedge}(k)\, e^{ikx}; \qquad S_{n+1}(f') = \sum_{-n}^{n} f'^{\wedge}(k)\, e^{ikx} \qquad (n \in \mathbb{N})$$

$$f'^{(-r)} = \sum{}' (ik)^{-r} f'^{\wedge}(k)\, e^{ikx} \qquad (0 < r \in \mathbb{N}),$$

where the sum $\sum'$ is extended over all integers except zero. Then $(f'^{(-r)})^{(r)} = f' - f'^{\wedge}(0)$, and $\| f' \|_{BV_{2\pi}^{(-k)}} = | f'^{\wedge}(0) | + \| f'^{(-k)} \|_{BV_{2\pi}}$ is an equivalent norm on $(C_{2\pi}^{(k)})'$. Furthermore, $(T_n, C_{2\pi})^{\perp} = \{f' \in BV_{2\pi} : f'^{\wedge}(k) = 0,\ | k | < n\}$, and $\| f' \|_{(T_n,C_{2\pi})} = \| S_n(f') \|_{(T_n,C_{2\pi})} = E_n{}'(S_n(f'); BV_{2\pi}) = \inf \| g \|_{BV_{2\pi}}$, where the infimum is taken over all $g \in BV_{2\pi}$ whose Fourier-coefficients of order up to n are equal to those of f'. (Note that in contrast to this, $E_n(f; C_{2\pi}) = \inf \| h \|_{C_{2\pi}}$, where the infimum is taken over all $h \in C_{2\pi}$ whose Fourier-coefficients of order $\geqslant n$ are equal to those of f!).

Observing these various facts (see Scherer [12]) and relations (7), (16), we can formulate as a consequence of Theorem III, case $q = 1$:

THEOREM IV. *Let f be a 2π-periodic distribution, and let α, r, k, l be as in Theorem* I. *The following assertions are equivalent*:

$$\sum_{n=1}^{\infty} [n^{-\alpha-r} E_n'(S_n(f'); BV_{2\pi})] \frac{1}{n} < \infty; \tag{IVa}$$

$$\sum_{n=1}^{\infty} [n^{k-\alpha-r} \| f'^{(-k)}_{C_{2\pi},n} \|_{BV_{2\pi}}] \frac{1}{n} < \infty; \tag{IVb}$$

$$f' = g^{(l)}, \qquad g \in BV_{2\pi} \quad \text{with} \quad \sum_{n=1}^{\infty} [n^{l-\alpha-r} \| g - f'^{(-l)}_{C_{2\pi},n} \|_{BV_{2\pi}}] \frac{1}{n} < \infty; \tag{IVc}$$

$$f' = h^{(r)}, \qquad h \text{ belonging to the dual of } lip^* \alpha, \qquad 0 < \alpha < 2. \tag{IVd}$$

The functions $f'_{C_{2\pi},n} \in BV_{2\pi}$ satisfy the conditions $f'_{C_{2\pi},n}{}^{\wedge}(k) = f'^{\wedge}(k)$ for $|k| < n$ and $\|f'_{C_{2\pi},n}\|_{BV_{2\pi}} = E_n'(S_n(f'); BV_{2\pi})$.

Such functions exist in view of the Hahn-Banach theorem. In view of (IVd), the case $r = 0$ of this theorem gives approximation theoretical characterizations of the dual of lip* α in the sense of the theorems of Jackson, Bernstein, Stečkin and Zamansky. Characterizations of a very different type may be found in De Leeuw [9].

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